Math 429 - Exercise Sheet 5

1. Express the adjoint representation of $\mathfrak{sl}_{2,\mathbb{C}}$ in terms of the irreducible representations $L(n)$ from the Lecture (try doing so as explicitly as possible).

2. Recall the isomorphism $\mathfrak{sl}_{2,\mathbb{C}} \cong \mathfrak{so}_{3,\mathbb{C}}$ from the previous exercise sheet. Express the standard 3-dimensional representation of $\mathfrak{sl}_{3,\mathbb{C}}$ in terms of the irreducible representations $L(n)$ from the Lecture (try doing so as explicitly as possible).

Note, however, that the isomorphism $\mathfrak{sl}_{2,\mathbb{C}} \cong \mathfrak{so}_{1,3}$ of real Lie algebras gives rise to a 4-dimensional real representation of $\mathfrak{sl}_{2,\mathbb{C}}$ which does not fit into the framework from Lecture, since the latter only applies to complex representations.

In what follows, we will show that any finite-dimensional complex representation $\mathfrak{sl}_{2,\mathbb{C}} \curvearrowright V$ is completely reducible, as long as H acts on V by a diagonalizable matrix (the latter condition holds due to the Jordan decomposition in semisimple Lie algebras, which we will study in Lecture 8).

3. Show that V is isomorphic (as a representation of $\mathfrak{sl}_{2,\mathbb{C}}$) to the direct sum of the generalized eigenspaces of the Casimir operator C

$$V = \bigoplus_{n \geq 0} \left\{ v \in V \,\middle|\, \left(C - \frac{n(n+2)}{2} \cdot I \right)^N (v) = 0 \text{ for some } N \gg 0 \right\}$$

As a consequence, we henceforth restrict attention to proving the complete reducibility of a representation $\mathfrak{sl}_{2,\mathbb{C}} \curvearrowright V$ on which C has a single generalized eigenvalue, say $\frac{n(n+2)}{2}$.

4. Show that any irreducible sub or quotient representation of V as above is isomorphic to $L(n)$, hence the eigenvalues of H (which we assume to be diagonalizable) are $n, n-2, \dots, 2-n, -n$.

5. Show that V is completely reducible by induction on $\dim V$ and the following statement: any surjective $\mathfrak{sl}_{2,\mathbb{C}}$ intertwiner $g : V \twoheadrightarrow L(n)$ splits, i.e. $\exists \psi : L(n) \to V$ such that $g \circ \psi = \text{Id}$.

Hint: note that $\text{Ker } g \cong L(n)^{\oplus k}$ *for some k. It suffices to pick some eigenvector of H in $V \setminus \text{Ker } g$ with eigenvalue n, and to show that it generates a subrepresentation of V isomorphic to $L(n)$.*